MODIFICATION OF THE ITERATION ALGORITHM FOR SOLVING THE INVERSE HEAT CONDUCTION PROBLEM

S. V. Mavrin

UDC 536.24.02

A modification of the iteration algorithm for solving the inverse heat conduction problem is proposed by the example of solution of the coefficient inverse heat conduction equation. In order to recover the functional dependence in the iteration algorithm of the gradient type it is proposed to determine separately the depth of descent for each of the components of the sought vector of parameters upon minimization of the mean-square error, which makes unnecessary solution of the conjugated boundary problem and computing of the derivatives of the target functional. The modification reduces the amount of computations and improves convergence of the algorithm.

Presently, methods of solving inverse problems find wide applications in experimental data processing. Rather frequently inverse problems are formulated in an extremum statement, and the iteration regularization method [1, 2] is used for their solution. In these algorithms three boundary problems should be solved at each iteration: the direct problem, the problem for the increment of the temperature field, and the conjugated equation. In addition, derivatives of the target functional are calculated based on the solution of the conjugated equation. Moreover, derivation of the conjugated equation and derivatives of the target functional are one of the most complex steps in derivation of the expressions for computations on which the iteration process is based.

In the present work an iteration algorithm is proposed in which the derivatives of the target functional are not used. As it will be shown in what follows, this makes it possible to improve convergence of the algorithm and simplify substantially the iteration process.

Let us consider the inverse heat conduction problem in the following formulation [1]. Determine the functions $T(x, \tau)$ and $\lambda(T)$ from the conditions:

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right), \quad 0 < x < b, \quad 0 < \tau < \tau_m, \tag{1}$$

$$T(x, 0) = T_0, \quad 0 \le x \le b, \tag{2}$$

$$T(0, \tau) = \varphi_1(\tau), \tag{3}$$

$$T(b, \tau) = \varphi_2(\tau), \tag{4}$$

$$T(d, \tau) = f(\tau), \quad 0 < d < b, \tag{5}$$

where functions $c(T)$, $\varphi_1(\tau)$, $\varphi_2(\tau)$, and $f(\tau)$, and the quantity T_0 are considered to be given. It is required to find $\lambda(T)$ from the conditions of minimum mean-square error

$$J = \int_0^{\tau_m} \left[T(d, \tau, \lambda(T)) - f(\tau)\right]^2 d\tau \to \min. \tag{6}$$

All-Russian Scientific Research Institute of Experimental Physics, Arzamas, Russia. Translated for Inzhenerno-Fizicheskii Zhurnal, Vol. 68, No. 3, pp. 494-499, May-June, 1995. Original article submitted June 4, 1993.

The existence of a solution of the problem and its uniqueness are shown in [1, 2]. We represent the sought function $\lambda(T)$ as a cubic B-spline:

$$\lambda(T) = \sum_{k=0}^{m+1} b_k B_k(T).$$

Thus, the problem of determination of the function $\lambda(T)$ is reduced to determination of the vector $b = \{b_0, b_1, ..., b_{m+1}\}$ that minimizes the functional (6) with constraints (1)-(5) imposed.

In accordance with the algorithm proposed in [1, 2] the boundary problem (1)-(5) is represented as a problem of heating of a bilayer plate; and the boundary problem for the increment of the temperature field $\Theta(x, \tau)$ and the boundary problem for the conjugated variable $\psi(x, \tau)$ are introduced. Using the conjugated variable we find components of the gradient vector of the target functional J_k, $k = 0, 1, ..., m+1$. In the case when the method of steepest descent is used for minimization of the target functional (6) the iteration process is built according to the expressions:

$$b_k^{p+1} = b_k^p - \alpha^p J_k^{'(p)}, \quad k = 0, 1, ..., m+1, \quad p = 0, 1, ...$$

where the slope depth α^p is determined from the condition

$$\min_{\alpha} J(b^p - \alpha^p J_k^{'(p)}). \tag{7}$$

Thus, in order to obtain expressions for computation one should carry out the following analytical transformations: to derive the boundary problem for the increment of the temperature field, the conjugated boundary problem, and the expression for derivatives of the target functional. The method of derivation of boundary problems and expressions for derivatives of the target functional is rather complicated, and the mathematical treatment is extremely cumbersome.

The results of computations show that upon recovery of functions the value of which is changed substantially within the range of definition, the slope depth α can take values close to zero which leads to a low convergence rate of the algorithm, and in certain cases even to its termination at values of the vector too far from the solution (in this case the value of the target functional (b) is intolerably high). The same problem exists when the method of conjugated gradients is used.

To overcome the problems outlined we will build an approximation of the vector b_n, $n = 0, 1, ..., m+1$ according to the expressions:

$$b_n^{p+1} = b_n^p - \alpha_n^p G_n^p, \quad n = 1, 2, ..., m, \tag{8}$$

$$b_0^{p+1} = 2b_1^{p+1} - b_2^{p+1}, \tag{9}$$

$$b_{m+1}^{p+1} = 2b_m^{p+1} - b_{m-1}^{p+1}, \tag{10}$$

where G_n^p is the descent direction.

We expand the increment of the temperature field $\Theta(\Delta b)$ into a Taylor series and restrict ourselves to linear terms in the expansion:

$$\Theta(\Delta b) = \sum_{n+1}^{m} \frac{\partial T}{\partial b_n} \Delta b.$$

Using the expression (8) we can write

$$\Theta(\Delta b) = -\sum_{n=1}^{m} \alpha_n \frac{\partial T}{\partial b_n} G_n.$$

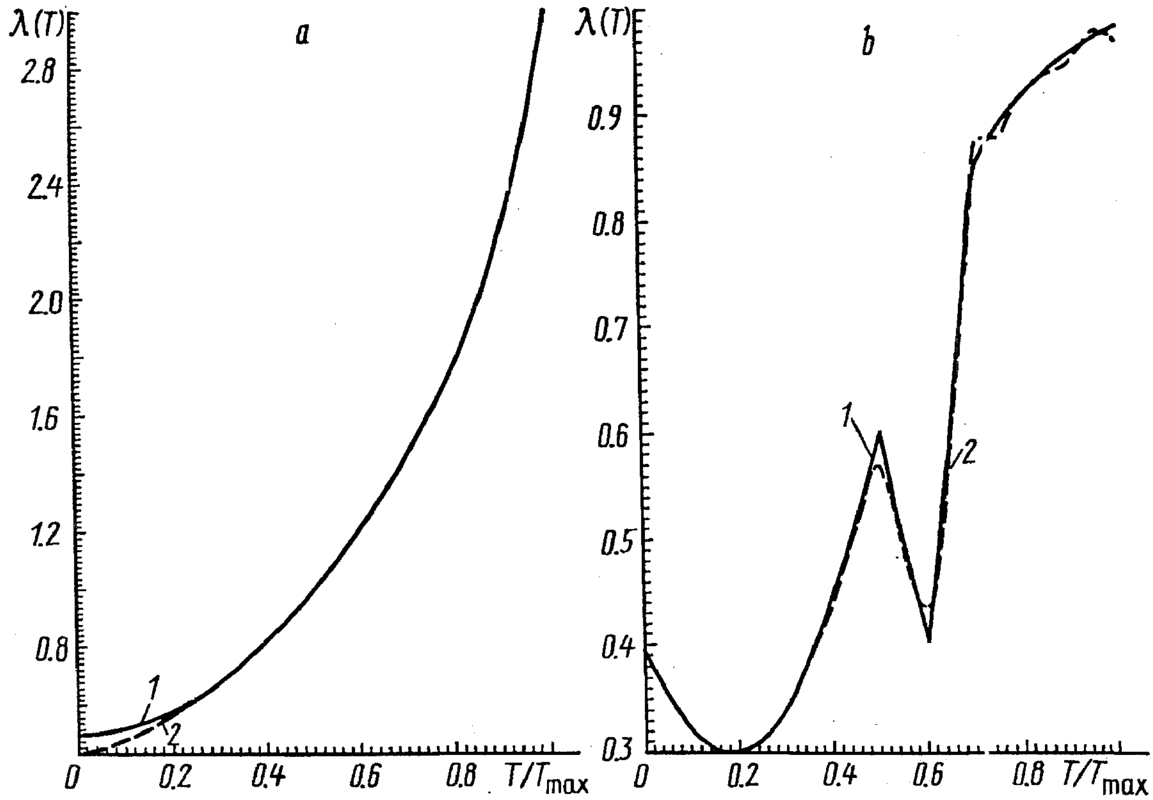


Fig. 1. Recovery of the heat conduction coefficient: 1) exact values; 2) recovered values. $T/T_{\max}$, $\lambda(T)$, dimensionless.

Setting $G_n = 1$, $n = 1, 2, \ldots, m + 1$ we finally obtain

$$\Theta(\Delta b) = -\sum_{n=1}^{m} \alpha_n \Theta_n ,$$

where Θ_n is the increment of the temperature field obtained from the solution of the boundary problem for the increment of the temperature field [3] at

$$\Delta b_k = \begin{cases} 0, & k \neq n; \\ 1, & k = n, \end{cases} \quad k = 1, 2, \ldots, m .$$

The value of the target functional at the $(p+1)$st iteration can be presented as follows:

$$J^{p+1} = \int_0^{\tau_m} \left[T(b^p) - \sum_{n=1}^{m} \alpha^p \Theta_n - f(\tau) \right]^2 d\tau .$$

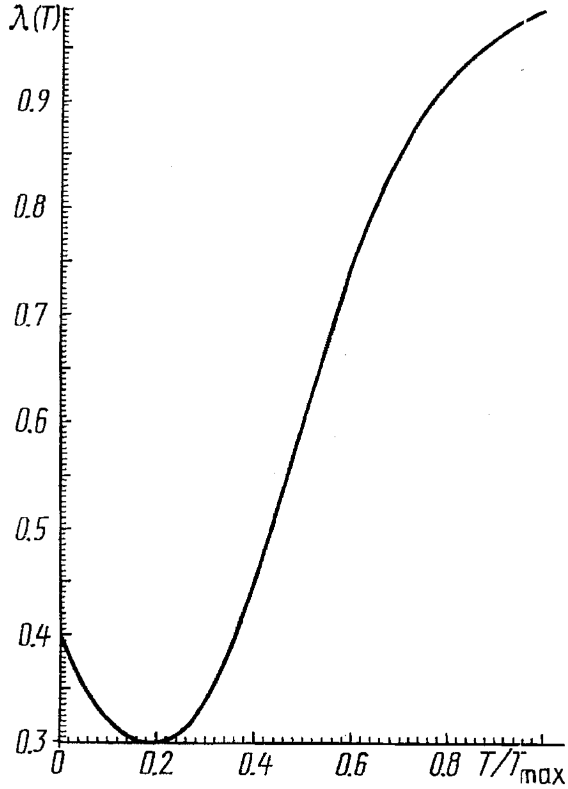
Minimizing this functional in accordance with (7) with respect to α_n, $n = 1, 2, \ldots, m$ we obtain a system of linear equations which in matrix form is as follows:

$$A\alpha = F , \tag{11}$$

where

$$a_{ij} = \int_0^{\tau_m} \Theta_i(d, \tau)\, \Theta_j(d, \tau)\, d\tau , \quad i, j = 1, 2, \ldots, m ,$$

$$f_i = -\int_0^{\tau_m} \left[T(d, \tau, b^p) - f(\tau) \right] \Theta_i(d, \tau)\, d\tau , \quad i = 1, 2, \ldots, m .$$



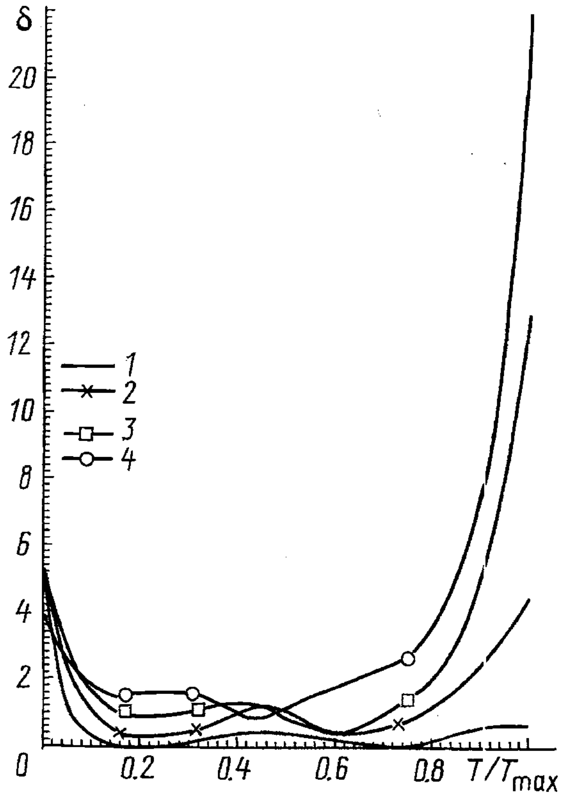


Fig. 2. The form of the function $\lambda(T)$.

Fig. 3. Mean error δ, %, of recovery of the heat conduction coefficient $\lambda(T)$ for various noise levels: 1) $\Delta = 0$; 2) 0.5%; 3) 1.5%; 4) 2.5%.

By solving system (11) we obtain the values α_n, $n = 1, 2, \ldots, m$.

Thus, at each of the iteration steps, m boundary problems for the increment of the temperature field are solved, and then system (11) is solved, and a new approximation of the vector b is found according to the expressions (8)-(10). The iteration process terminates according to one of the criteria proposed in [1, 2]. Compared to the algorithms described in [1, 2], the execution time of each of the iteration steps is increased, but the reduction of the number of iterations leads to a decrease in the total time of solution of the problem.

To compare the computational properties of the algorithm proposed with the algorithms proposed in [1, 2] we solved the problem presented in [3]. As in [3], the boundary problems were solved using an inexplicit difference scheme [4]. When solving the direct problem, iterations over coefficients were carried out. In [3] the problem was solved after 25 iterations. When the algorithm proposed is used already at the sixth iteration the increment of the target functional was 1%, whereas after the eighth iteration the increments of the target functional and the vector b were not changed. The results of the recovery of the dependence $\lambda(T)$ are presented in Fig. 1a. In [3] 75 boundary problems were solved after 25 iterations. In the algorithm proposed one boundary problem for calculation of the field $T(x, \tau)$ and four problems for the increment of the temperature field Θ_n, $n = 1, 2, 3, 4$ were solved. By this means, 40 boundary problems were solved after eight iterations which is almost two times less than in [3].

We also carried out the recovery of the nonmonotonic function $\lambda(T)$ with discontinuities in the first and second derivatives. The initial data were obtained using the same method as in [3]. The number of the reference sensors was nine. The exact and recovered values of the function $\lambda(T)$ are presented in Fig. 1b. To determine the effect of the errors in temperature measurements, positions of the thermosensors, and calculations on the result of solution of the inverse problem the problem of the recovery of the undulatory function $\lambda(T)$ presented in Fig. 2 was solved. The errors were modeled by imposing random uniformly distributed interference on the boundary temperatures and temperature at the internal points. The interference was calculated according to the formula

$$T_i^* = \frac{\Delta}{100} W_i T_i,$$

where T_i is the temperature at the i-th site of the time lattice, W_i is a random number uniformly distributed within the interval $[-1, 1]$; Δ is the noise level; T_i^* is the additive interference at the i-th site of the time lattice. As in the preceding example, the number of the reference thermosensors was equal to nine. As the calculations have shown, an increase in the number of knots of interpolation of the function $\lambda(T)$ over eight does not lead to a noticeable decrease in the error of the recovery of the function. Therefore it was taken to be eight. The error of recovery was calculated according to the expression

$$\delta\lambda_i = \frac{\lambda_i^{\text{rec}} - \lambda_i^{\text{ex}}}{\lambda_i^{\text{ex}}} 100\% , \quad i = 1 , 2 , \dots , m ,$$

where λ_i^{rec} is the value of the recovered function $\lambda(T)$ at the i-th knot of interpolation; λ_i^{ex} is the exact value of the function $\lambda(T)$ at the i-th knot of interpolation; $\delta\lambda_i$ is the error of the recovery of the function $\lambda(T)$.

The noise level was taken to be 0, 0.5, 1.5, and 2.5%. The result of solving a poorly formulated problem depends on the specific realization of the random sequence which models the noise. Therefore Fig. 3 presents plots of the mean error of recovery of the function at various noise levels. The mean error was determined in the following manner. For each of the noise levels the inverse problem was solved eight times. For each of these eight problems separate realizations were given of the random sequence which models noise. By this means, for each of the knots of the interpolation function $\lambda(T)$ eight recovery errors were determined. Then for each of the interpolation knots the mean value of the error was calculated.

It is clear from Fig. 3 that, with the exception of the edge of the temperature interval, the error weakly depends on the noise level. Thus, when employing in practice the method described it can be recommended either to carry out smoothing (filtration) of experimental curves, or simply not to use the values of the sought function at the edge of the temperature interval.

As is clear from the results presented, the proposed modification of the algorithm leads to its substantial simplification, reduces the calculation time, and makes it possible to recover rather complex functions. The level of various interferences virtually does not affect the accuracy of the solution of the inverse problem. The algorithm can be implemented for the solution of other inverse problems under extreme conditions.

NOTATION

$C(T)$, volume heat capacity; T, temperature; τ, time; τ_m, right boundary value of the time interval; x, coordinate; b, right boundary value of the space interval; $\lambda(T)$, heat conduction coefficient; $\varphi(\tau)$, boundary temperature; J, target functional; B, spline; b_k, $k = 0, 1, \dots, m + 1$, parameter of the spline approximation of the function $\lambda(T)$; p, iteration number; J', gradient of the target functional; α, step length in the method of steepest descent; G, parameter of the method of steepest descent; $\theta(x, \tau)$, increment of the temperature field; $\delta\lambda_i$, $i = 1, 2, \dots, m$, error of the recovery of the function $\lambda(T)$ at the i-th interpolation knot; Δ, noise level.

REFERENCES

1. O. M. Alifanov, E. A. Artyukhin, and S. V. Rumyantsev, Extremum Methods of Solution of Poorly Formulated Problems [in Russian], Moscow (1983).
2. O. M. Alifanov, Inverse Heat Conduction Problems [in Russian], Moscow (1988).
3. E. A. Artyukhin, Inzh.-Fiz. Zh., **41**, No. 4, 587-592 (1981).
4. A. A. Samarskii, Theory of Difference Schemes [in Russian], Moscow (1977).